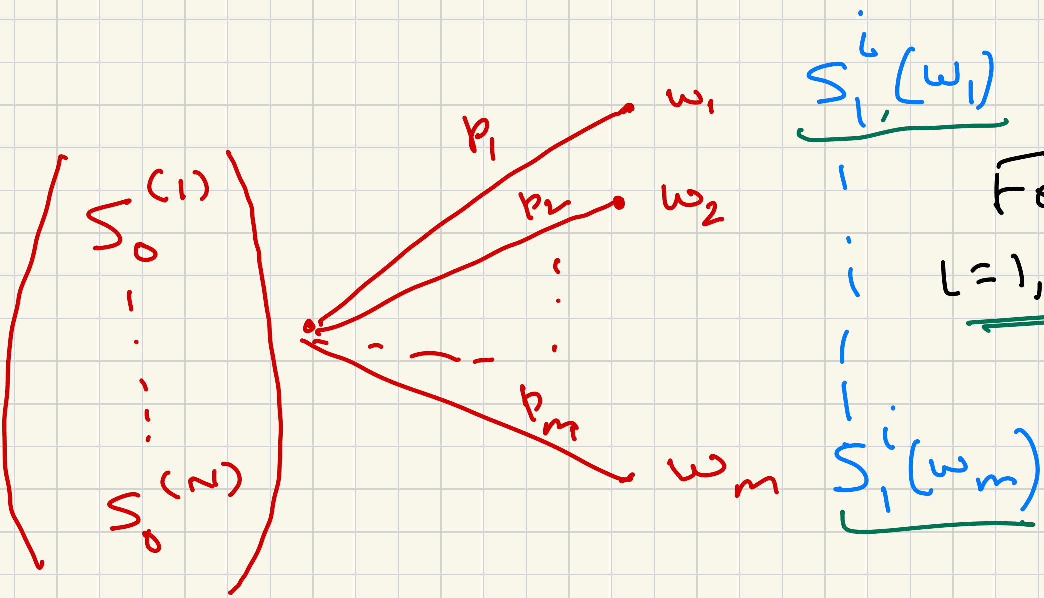
Consider 2 time period model with N securities and m scenarios



For $i = 1, .., N$

Each $p_i > 0$.

Let

$$S_1 = \begin{pmatrix} S^{(1)}(\omega_1) & \cdots & S^{(N)}(\omega_1) \\ S^{(1)}(\omega_2) & \cdots & S^{(N)}(\omega_2) \\ \vdots & & \\ S^{(1)}(\omega_m) & \cdots & S^{(N)}(\omega_m) \end{pmatrix}$$

subscript 1 is hidden

Arbitrage: A trading strategy $\theta \in \mathbb{R}^N$:

$$S_0^T \theta < 0 \ \& \ S_1 \theta = 0$$

or

$$S_0^T \theta \leq 0 \ \& \ S_1 \theta \geq 0, \ S_1 \theta \neq 0$$

Elementary security is of the form

$$e_j = (0, 0, \ldots, 1, 0, \ldots 0)$$

position j.

$j = 1, \ldots, m$.

A security X or a contingent claim is said to be attainable if $\exists$ a trading strategy

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix} :$$

$$\downarrow$$

$$\begin{bmatrix} X(\omega_1) \\ X(\omega_2) \\ \vdots \\ \vdots \\ X(\omega_m) \end{bmatrix} = \begin{bmatrix} S^{(1)}(\omega_1) & & S^{(N)}(\omega_1) \\ \vdots & & \\ \vdots & & \\ S^{(1)}(\omega_m) & \cdots & S^{(N)}(\omega_m) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

i.e., $X = S\Theta$

e.g.

$$\begin{bmatrix} 1.0194 \\ 3.4045 \\ 2.497 \end{bmatrix}$$

$w_1 \; [1.03, 3, 2]$

$w_2 \; [1.03, 4, 1]$

$w_3 \; [1.03, 2, 4]$

$w_4 \; [1.03, 5, 2]$

$$X = \begin{bmatrix} 7.47 \\ 6.97 \\ 9.97 \\ 10.47 \end{bmatrix}$$ attainable at $$\Theta = \begin{bmatrix} -1 \\ 1.5 \\ 2 \end{bmatrix}$$

We say that a vector

$$\begin{bmatrix} \pi_1 \\ \vdots \\ \pi_m \end{bmatrix} > 0$$

is a vector of state prices if the $t=0$ price, P, of any attainable security, X, satisfies

$$P = \sum_{k=1}^{m} \pi_k X(\omega_k)$$

Then Π_k denotes state k price.

To simplify analysis we assume existence of a risk free asset that returns $(1+r)$ at time 1 in every scenario for a rupee invested at time 0. (One of the N assets)

An equivalent martingale measure or risk neutral probability measure is a set of probabilities

$$Q = (q_1, q_2, \ldots, q_m) \geq 0$$

such that deflated security prices are martingales.

That is,

$$\underline{\bar{S}_0^{(i)}} = \frac{S_0^{(i)}}{1} = E_Q\left[\frac{S^{(i)}}{1+r}\right]$$

$$= E_Q \, \underline{\bar{S}}^{(i)}$$

$$\Rightarrow \boxed{\bar{S}_0^{(i)} = \sum_{j=1}^{m} \bar{S}^{(i)}(\omega_j)\, q_j}$$

for all $i = 1, \ldots, N$.

Result: A set of positive state prices exists iff risk neutral measure exists

State price π exists

Then
$$S_0^{(i)} = \sum_{k=1}^{m} \pi_k S^{(i)}(\omega_k) \quad \forall i.$$

Since risk free security exists

$$\Rightarrow \quad 1 = \sum_{k=1}^{m} \pi_k (1+r)$$

$$\Rightarrow \quad \sum_{k=1}^{m} \pi_k = (1+r)^{-1}$$

$$\Rightarrow \quad \bar{S}_0^{(i)} = \sum_{k=1}^{m} \frac{\pi_k}{\sum_{j=1}^{m} \pi_j} \cdot \frac{S^{(i)}(\omega_k)}{1+r}$$

$$\Rightarrow \quad \bar{S}_0^{(i)} = \sum_{k=1}^{m} q_k \, \bar{S}^{(i)}(\omega_k) \quad \forall i \qquad (1)$$

So $q_k = \dfrac{\pi_k}{\sum_{j=1}^{m} \pi_j}$, $k = 1, \ldots, m$

is the risk neutral measure.

Now if (1) holds

$$\Rightarrow \quad S_0^{(i)} = \sum_{k=1}^{m} \pi_k \, S^{(i)}(\omega_k)$$

where $\pi_k = \dfrac{q_k}{(1+r)}$. $\forall k$.

Result: There is no-arbitrage if and only if there exists a set of strictly positive state prices.

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Recall that an arbitrage exists if $\exists$ a portfolio $\theta \in \mathbb{R}^n$:

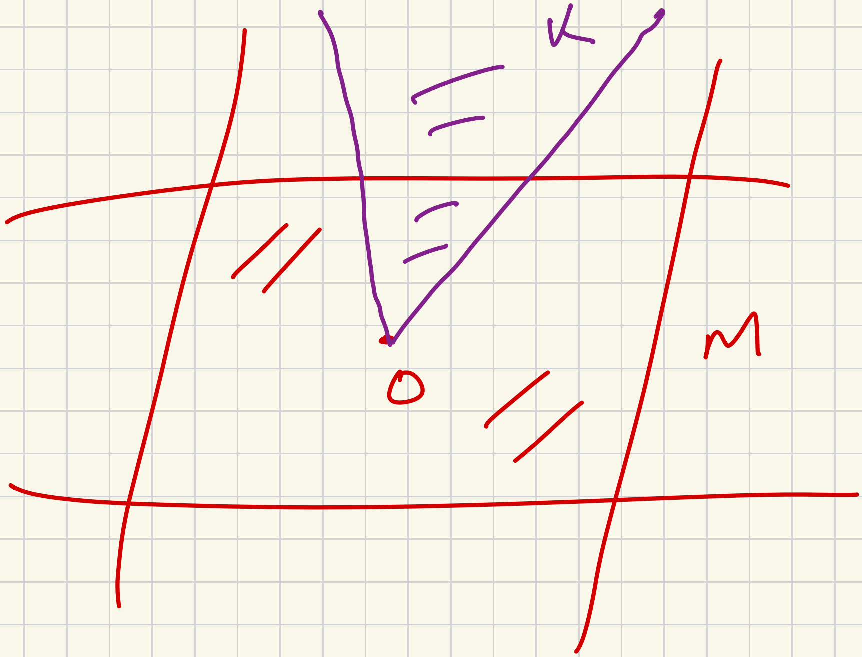
$$S_0^T \theta < 0 \quad \& \quad S_1 \theta = 0$$

or

$$S_0^T \theta \leq 0 \quad \& \quad S_1 \theta \geq 0, \; S_1 \theta \neq 0.$$

Consider $M = \{ (-S_0^T \theta, S_1 \theta) : \theta \in \mathbb{R}^N \}$

$K = \{ \mathbb{R}^+ \times \mathbb{R}^m_+ \}$



K is a cone

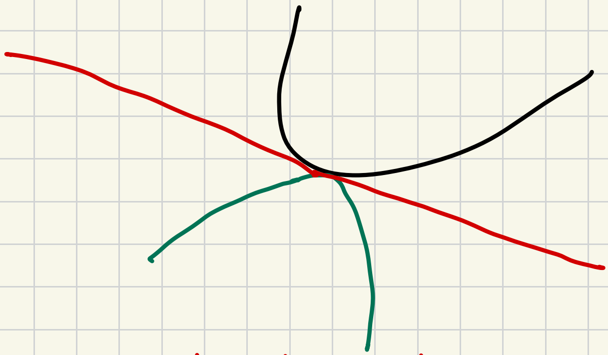
$x \in K \Rightarrow \lambda x \in K$ for any $\lambda > 0$

M is a subspace

$x, y \in M \Rightarrow \alpha x + \beta y \in M$

$\forall \alpha, \beta$.

& K & M are convex sets.



Separating hyperplane theorem ensures that there exist an

$a \in \mathbb{R}^{m+1}$:

$a^T x > a^T y$

$\forall\ x \in K$ and non-zero.

$\forall\ y \in M$.

But M is a subspace

Must be that $a^T y = 0$
$\forall\ y \in M$.

$\Rightarrow \quad a^T x > 0 \quad \forall\ x \in K,\ x \neq 0.$

$$\Rightarrow \quad a_1 x_1 + (a_2, \ldots, a_{m+1})^T (x_2, \ldots, x_{m+1}) > 0$$
$\forall\ x \in K,\ x \neq 0.$

$\Rightarrow$ each $a_i > 0$

Now

$$a_1(-s_0^T \theta) + (a_2, \ldots, a_{m+1})^T (s_1 \theta) = 0$$
$\forall\ \theta$.

$$\left(-s_0^T + \left(\frac{a_2}{a_1}, \ldots, \frac{a_{m+1}}{a_1}\right)^T s_1\right)\theta = 0$$
$\forall\ \theta$

$\Rightarrow \quad S_0 = S_1^T \pi$ for some

$\pi > 0$

$\Rightarrow$ state price vector exists.

Now suppose positive state vector π exists.

$$S_0 = S_1^T \pi$$

$$S_0^T \theta = \pi^T S_1 \theta$$

If $S_1 \theta > 0 \Rightarrow S_0^T \theta > 0$

If $S_1 \theta = 0 \Rightarrow S_0^T \theta = 0$

$\Rightarrow$ No arbitrage.

In summary

Result: Assume there exists a risk free security.

The following are equivalent

1) Absence of arbitrage
2) Existence of strictly positive state price process
3) Existence of EMM

Complete markets

If every random variable X is attainable then the market is complete. Else it is incomplete.

That is in complete mkts

$\forall X \in \mathbb{R}^m, \exists \theta \in \mathbb{R}^N$:

$$S_1 \theta = X$$

where

$$S_1 = \begin{pmatrix} S^{(1)}(\omega_1) & \cdots & S^{(N)}(\omega_1) \\ \vdots & & \vdots \\ S^{(1)}(\omega_m) & & S^{(N)}(\omega_m) \end{pmatrix}$$

$N \geq m$ & S_1 has rank m

Result: Assume $\exists$ a security with strictly +ve price process and there are no arbitrage opportunities.

Then market is complete if and only if there exists exactly one E.M.M.

Suppose market is complete.

$\Rightarrow$ Unique positive state price process, since elementary securities have unique positive price

$\Rightarrow$ unique E. m. m.

Now suppose unique E.m.m. but market is not complete

We show that this leads to a contra-
-diction.

Incompleteness $\Rightarrow$
$\exists$ X that cannot be attained.

$\Rightarrow$ There does not

exist θ : $S\theta = X$

$\Rightarrow$ X does not lie in span of S.

$\Rightarrow$ ' $\exists\, h$: $h^T S = 0$

& $h^T X > 0$. '

let Q be EMM

Set

$$\bar{q}(\omega_j) = q(\omega_j) + \lambda h_j (1+r)$$

Now

$$\sum_{j=1}^{m} h_j(1+r) = 0$$

Choose λ sufficiently small so that $\hat{q}(\omega_j) > 0 \ \forall j$. & $\hat{q}$ is a probability vector.

$$S_0^{(j)} = \sum_{k=1}^{m} \frac{S^{(j)}(\omega_k)}{(1+r)} \hat{q}(\omega_k)$$

$$= \sum_{k=1}^{m} \frac{S^{(j)}(\omega_k)}{1+r} \left(q(\omega_k) + \lambda h_j (1+r)\right)$$

Since $\sum_{k=1}^{m} S^{(j)}(\omega_k) h_j = 0 \quad \forall j.$

So

$$S_0^{(i)} = \sum_{k=1}^{m} \overline{S^{(j)}(\omega_k)}\, \tilde{q}(\omega_j)$$

$\Rightarrow$ $\exists$ mm is not unique.

A contradiction!